Proyecciones Journal of Mathematics
Vol. 40, N^o 5, pp. 1097-1116, October 2021.
Universidad Católica del Norte
Antofagasta - Chile
10.22199/issn.0717-6279-4258

k-super cube root cube mean labeling of graphs

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Received : June 2020. Accepted : January 2021

Abstract

Consider a graph G with $|V(G)| = p$ and $|E(G)| = q$ and let $f : V(G) \rightarrow \{\mathrm{k}, \mathrm{k}+1, \mathrm{k}+2, \ldots \mathrm{p}+\mathrm{q}+\mathrm{k}-1\}\}$ be an injective function. The induced edge labeling f^* for a vertex labeling f is defined by $f^*(e) = \left\lfloor \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rfloor$ or $\left\lceil \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rceil$ for all $e = uv \in E(G)$ is bijective. If $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{\mathrm{k}, \mathrm{k}+1, \mathrm{k}+2, \ldots, p+q+k-1\}$, then f is called a k-super cube root cube mean labeling. If such labeling exists, then G is a k-super cube root cube mean graph. In this paper, I introduce k-super cube root cube mean labeling and prove the existence of this labeling to the graphs viz., triangular snake graph T_n, double triangular snake graph $D(T_n)$, Quadrilateral snake graph Q_n, double quadrilateral snake graph $D(Q_n)$, alternate triangular snake graph $A(T_n)$, alternate double triangular snake graph $AD(T_n)$, alternate quadrilateral snake graph $A(Q_n)$, & alternate double quadrilateral snake graph $AD(Q_n)$.

Keywords: *k-super cube root cube mean labeling, k-super cube root cube mean graph, snake graph, alternate snake graph.*

MSC(2020): *05C78.*

1. Introduction

In this paper, all graphs are simple, finite and undirected with $|V(G)| = p$ and $|E(G) = q$. Labeling of a graph is an assignment of integers to the vertices or edges or both subject to certain conditions. For detailed study of graph labeling, refer to J. A. Gallian [2]. Many researchers in practice contributed various types of labeling like k-super mean labeling [3,7], root square mean labeling [6], k-super root square mean labeling [1], cube root cube mean labeling [4], super cube root cube mean labeling [5], etc. In this paper, I have added another type of graph labeling, i.e., k-super cube root cube mean labeling. Consider a graph G with $|V(G)| = p$ and $|E(G)| = q$ and let $f : V(G) \to \{\mathrm{k},\ \mathrm{k}+1,\ \mathrm{k}+2, \ldots,\ \mathrm{p}+\mathrm{q}+\mathrm{k}-1\ \}$ be an injective function. The induced edge labeling f^* for a vertex labeling f is defined by $f^*(\mathrm{e}) = \left\lfloor \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rfloor$ or $\left\lceil \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rceil$ for all $e = uv \in E(G)$ is bijective. If $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{\mathrm{k},\ \mathrm{k}+1,\ \mathrm{k}+2, \ldots,\ \mathrm{p}+\mathrm{q}+\mathrm{k}-1\}$, then f is known as k-super cube root cube mean labeling. If such labeling exists, then G is a k-super cube root cube mean graph. In this paper, it is assumed that k is an integer and its value is ≥ 1.

2. Preliminaries

Definition 2.1. *The triangular snake graph T_n is obtained from a path P_n by replacing each edge of the path by a triangle C_3.That is a triangular snake graph is obtained from a path $u_1,\ u_2, \ldots, u_n$ by joining u_i and u_{i+1} to a new vertex $v_i, 1 \leq i \leq n-1$.*

Definition 2.2. *An alternate triangular snake graph $A(T_n)$ is obtained from a path $u_1,\ u_2,\ \ldots,\ u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i. That is every alternate edge of a path is replaced by a triangle C_3.*

Definition 2.3. *A double triangular snake graph $D(T_n)$ consists of two triangular snakes that have a common path.*

Definition 2.4. *An alternate double triangular snake graph $AD(T_n)$ consists of two alternate triangular snakes having a common path. That is to construct an alternate double triangular snake graph $AD(T_n)$, we have to* *join u_i and u_{i+1}, $1 \leq i \leq n-1$ (alternately) from a path with vertices u_1, u_2, ..., u_n to the vertices v_j and w_j, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.*

Definition 2.5. *A quadrilateral snake graph Q_n is obtained from a path P_n by replacing each edge of the path by a cycle C_4. That is a quadrilateral snake graph Q_n is obtained from a path u_1, u_2, ..., u_n by joining u_i, u_{i+1} to new vertices v_i and w_i respectively and adding edges $v_i w_i$ for $i = 1, 2, ..., n-1$.*

Definition 2.6. *To construct an alternate quadrilateral snake graph $A(Q_n)$, we have to join u_i and u_{i+1} alternately from a path with vertices u_1, u_2, ..., u_n to the vertices v_j, w_j respectively then joining v_j and w_j, $1 \leq i \leq n-1 \& 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. That is every alternate edge of a path is replaced by a cycle C_4.*

Definition 2.7. *A double quadrilateral snake graph $D(Q_n)$ is obtained from two quadrilateral snakes that have a common path.*

Definition 2.8. *An alternate double quadrilateral snake graph $AD(Q_n)$ is obtained from two alternative quadrilateral snakes that have a common path.*

3. Main Results

Theorem 3.1. *Any triangular snake graph T_n is a k-super cube root cube mean graph.*

Proof. Let T_n be a triangular snake graph.
Here p = 2n - 1 & q = 3n - 3
Hence p + q = 5n - 4.
Define a function $f : V(T_n) \rightarrow \{k, k+1, k+2, \ldots, p+q+k-1\}$ by
$f(u_i)$ = k + 5i - 5, $1 \leq i \leq n$
f(vi) = k + 5i -3, $1 \leq i \leq$ n - 1.
Then, the edge labels of T_n are
$f^*(u_i v_i)$ = k + 5i - 4, $1 \leq i \leq$ n - 1
$f^*(u_i u_{i+1})$ = k + 5i - 2, $1 \leq i \leq$ n - 1
$f^*(u_{i+1} v_i)$ = k + 5i -1, $1 \leq i \leq$ n - 1.
Hence $f(V(T_n)) \cup \{f^*(e) : e \in E(T_n)\} = \{k, k+1, k+2, \ldots, p+q+k-1\}$.
Therefore any Triangular snake graph T_n is a k-super cube root cube mean graph.

An example of 400-super cube root cube mean labeling of T_5 is shown in Figure 1

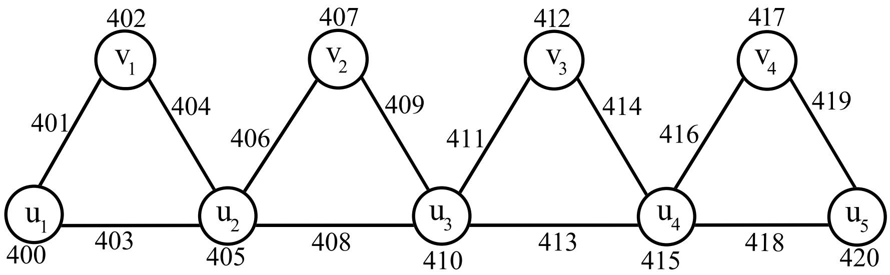

Figure 1: 400-super cube root cube mean labeling of T_5

□

Theorem 3.2. *Any alternate triangular snake graph $A(T_n)$ is a k-super cube root cube mean graph.*

Proof. Let $A(T_n)$ be an alternate triangular snake graph.
In this theorem, consider two cases.

Case 1. The triangle in $A(T_n)$ starts from u_1
In this case

$$p=\begin{cases} \frac{3n}{2}, & if\ n\ is\ even; \\ \frac{3n-1}{2}, & if\ n\ is\ odd. \end{cases}$$

$$\&\ q=\begin{cases} 2n-1, & if\ n\ is\ even; \\ 2n-2, & if\ n\ is\ odd. \end{cases}$$

$$\text{Hence } p+q=\begin{cases} \frac{7n-2}{2}, & if\ n\ is\ even; \\ \frac{7n-5}{2}, & if\ n\ is\ odd. \end{cases}$$

Define a function $f: V(A(T_n)) \to \{k,\ k+1,\ k+2, \dots,\ p+q+k-1\ \}$ by

$f(u_{2i-1}) = k + 7i - 7$, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n+1}{2}$ if n is odd.
$f(u_{2i}) = k + 7i - 2$, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

$f(v_i) = k + 7i - 5$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
Then, the edge labels of $A(T_n)$ are

$f^*(u_{2i-1}u_{2i}) = k + 7i - 4$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i}u_{2i+1}) = k + 7i - 1$, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i-1}v_i) = k + 7i - 6$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i}v_i) = k + 7i - 3$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
Hence $f[V(A(T_n))] \cup \{f^*(e) : e \in E(A(T_n))\} = \{k, k+1, k+2, \ldots, p+q+k-1\}$.
An example of 15-super cube root cube mean labeling of $A(T_8)$ [Triangle

start from u_1] is shown in Figure 2.

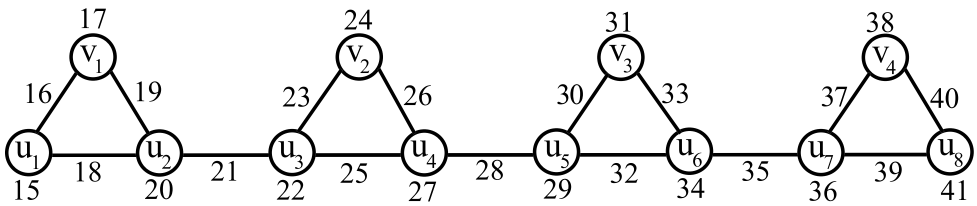

Figure 2: 15-super cube root cube mean labeling of $A(T_8)$ [Triangle start from u_1]

Case 2. The triangle in $A(T_n)$ starts from u_2

In this case $p = \begin{cases} \frac{3n-2}{2}, & if\ n\ is\ even; \\ \frac{3n-1}{2}, & if\ n\ is\ odd. \end{cases}$

& $q = \begin{cases} 2n-3, & if\ n\ is\ even; \\ 2n-2, & if\ n\ is\ odd. \end{cases}$

Hence $p + q = \begin{cases} \frac{7n-8}{2}, & if\ n\ is\ even; \\ \frac{7n-5}{2}, & if\ n\ is\ odd. \end{cases}$

Define a function $f : V(A(T_n)) \rightarrow \{k, k+1, k+2, \ldots, p+q+k-1\}$ by

$f(u_{2i-1}) = k + 7i - 7$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n+1}{2}$ if n is odd.
$f(u_{2i}) = k + 7i - 5$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

f(v_i) = k + 7i - 3, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
Then, the edge labels of A(T_n) are
$f^*(u_{2i-1}u_{2i})$ = k +7i - 6, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i}u_{2i+1})$ = k +7i - 2, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i}v_i)$ = k + 7i - 4, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i+1}v_i)$ = k + 7i - 1, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
Hence $f[V(A(T_n))] \cup \{f^*(e) : e \in E(A(T_n))\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.
An example of 15-super cube root cube mean labeling of $A(T_7)$ [Triangle start from u_2] is shown in Figure 3.

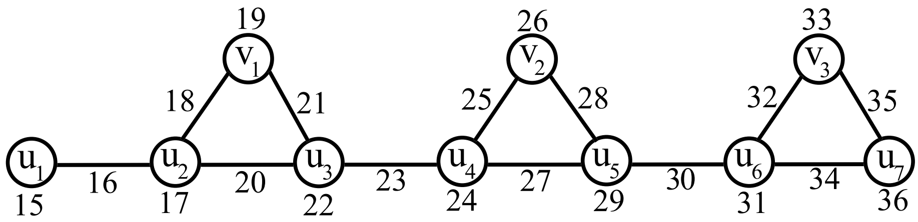

Figure 3: 15-super cube root cube mean labeling of $A(T_7)$ [Triangle start from u_2]

From the above cases, an alternate triangular snake graph $A(T_n)$ is a k-super cube root cube mean graph.

□

Theorem 3.3. *Any double triangular snake graph $D(T_n)$ is a k-super cube root cube mean graph.*

Proof. Let D(T_n) be a double triangular snake graph.
Here p = 3n - 2 & q = 5n - 5 Hence p + q = 8n - 7.
Define a function f:V(D(T_n)) → $\{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\ \}$ by
f(u_1) = k , for all k.

$$f(u_2) = \begin{cases} k+5, & k = 1,2,3,\ldots 10; \\ k+6, & otherwise. \end{cases}$$

$f(u_i) = k + 8i - 10,\ 3 \leq i \leq n$
$f(v_i) = k + 8i,\ \ 1 \leq i \leq n - 1$
$f(w_i) = k + 8i - 6,\ \ 1 \leq i \leq n - 1.$
Then, the edge labels of $D(T_n)$ are
$f^*(u_iu_{i+1}) = k + 8i - 5,\ 1 \leq i \leq n - 1.$

$$f^*(u_1v_1) = \begin{cases} k+6, & k = 1,2,3,\dots 10; \\ k+4, & otherwise. \end{cases}$$

$f^*(u_iv_i) = k + 8i - 4,\ 2 \leq i \leq n - 1$
$f^*(u_{i+1}v_i) = k + 8i -1,\ 1 \leq i \leq n - 1$
$f^*(u_iw_i) = k + 8i - 7,\ \ 1 \leq i \leq n - 1$

$$f^*(u_2w_1) = \begin{cases} k+4, & k = 1,2,3,\dots 10; \\ k+5, & otherwise. \end{cases}$$

$f^*(u_{i+1}w_i) = k + 8i -3,\ 2 \leq i \leq n -1.$
Hence $f[V(D(T_n))] \cup \{f^*(e) : e \in E(D(T_n))\} = \{k,\ k+1,\ k+2, \dots,\ p+q+k-1\}$.
Therefore any double triangular snake graph $D(T_n)$ is a k-super cube root

cube mean graph.
An example of 75-super cube root cube mean labeling of $D(T_5)$ is shown

in Figure 4.

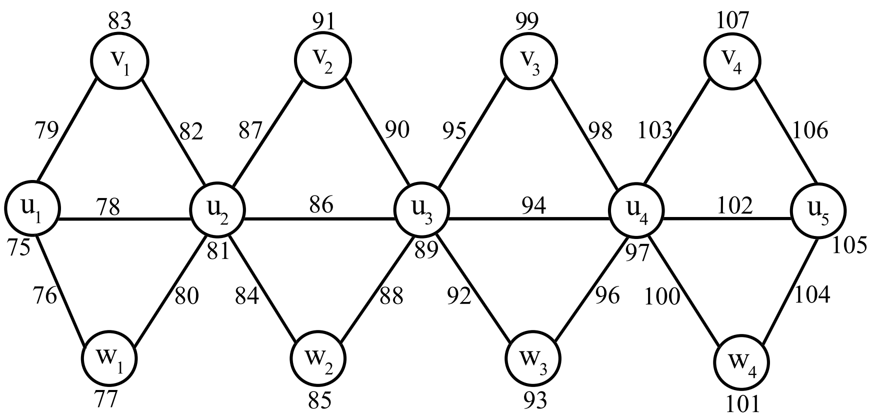

Figure 4: 75-super cube root cube mean labeling of $D(T_5)$

□

Theorem 3.4. *Any alternate double triangular snake graph $AD(T_n)$ is a k-super cube root cube mean graph.*

Proof. Let AD(Tn) be an alternate double triangular snake graph. In this theorem, consider two cases.

Case 1. The triangle in AD(T_n) starts from u_1

In this case p = $\begin{cases} 2n, & if\ n\ is\ even; \\ 2n-1, & if\ n\ is\ odd. \end{cases}$

& q = $\begin{cases} 3n-1, & if\ n\ is\ even; \\ 3n-3, & if\ n\ is\ odd. \end{cases}$

Hence p + q = $\begin{cases} 5n-1, & if\ n\ is\ even; \\ 5n-4, & if\ n\ is\ odd. \end{cases}$

Define a function f:V(AD(T_n))→ {k, k + 1, k + 2, . . . , p + q + k − 1 } by

f(u_{2i-1}) = k + 10i - 8, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n+1}{2}$ if n is odd.

f(u_{2i}) = k + 10i - 5, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f(v_i) = k + 10i - 2, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f(w_i) = k + 10i - 10, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

Then, the edge labels of AD(T_n) are

f*($u_{2i-1}u_{2i}$) = k +10i - 6, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f*($u_{2i}u_{2i+1}$) = k +10i - 1, $1 \le i \le \frac{n-2}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f*($u_{2i-1}v_i$) = k + 10i - 4, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f*($u_{2i}v_i$) = k + 10i - 3, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f*($u_{2i-1}w_i$) = k + 10i - 9, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

f*($u_{2i}w_i$) = k + 10i - 7, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

Hence f[V(AD(T_n))] ∪ {f*(e) : e ∈ E(AD(T_n))}={ $k,\ k+1,\ k+2, \ldots,\ p+q+k-1$}.

An example of 50-super cube root cube mean labeling of AD(T_8)[triangle start from u_1] is shown in Figure 5.

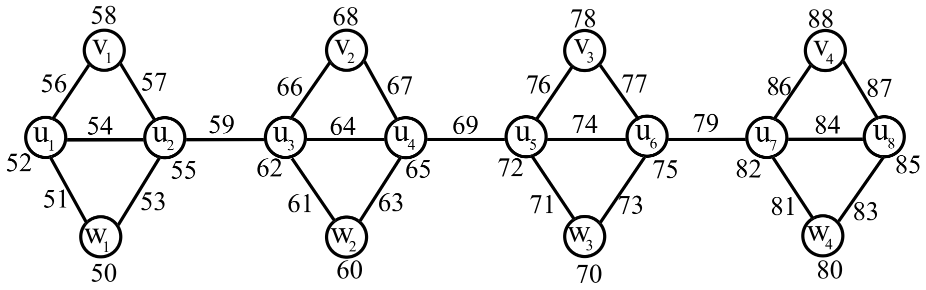

Figure 5: 50-super cube root cube mean labeling of $AD(T_8)$ [triangle start from u_1]

Case 2. The triangle in AD(T_n) starts from u_2

In this case p = $\begin{cases} 2n-2, & if\ n\ is\ even; \\ 2n-1, & if\ n\ is\ odd. \end{cases}$

& q = $\begin{cases} 3n-5, & if\ n\ is\ even; \\ 3n-3, & if\ n\ is\ odd. \end{cases}$

Hence p + q = $\begin{cases} 5n-7, & if\ n\ is\ even; \\ 5n-4, & if\ n\ is\ odd. \end{cases}$

Define a function f: V(AD(T_n)) $\rightarrow$ {k, k + 1, k + 2, . . . , p + q + k − 1 }

by
f(u_{2i-1}) = k + 10i - 10, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n+1}{2}$ if n is odd.
f(u_{2i}) = k + 10i - 8, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.
f(v_i) = k + 10i - 4, $1 \le i \le \frac{n-2}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.
f(w_i) = k + 10i - 5, $1 \le i \le \frac{n-2}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.
Then, the edge labels of AD(T_n) are
f*($u_{2i-1}u_{2i}$) = k +10i -9, $1 \le i \le \frac{n}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.
f*($u_{2i}u_{2i+1}$) = k +10i - 3, $1 \le i \le \frac{n-2}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.
f*($u_{2i}v_i$) = k + 10i - 6, $1 \le i \le \frac{n-2}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.
f*($u_{2i+1}v_i$) = k + 10i - 1, $1 \le i \le \frac{n-2}{2}$ if n is even& $1 \le i \le \frac{n-1}{2}$ if n is odd.
f*($u_{2i+1}w_i$) = k + 10i - 2, $1 \le i \le \frac{n-2}{2}$ if n is even & $1 \le i \le \frac{n-1}{2}$ if n is odd.

$f^*(u_{2i}w_i)$ = k + 10i - 7, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
Hence $f[V(AD(T_n))] \cup \{f^*(e) : e \in E(AD(T_n))\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.
An example of 50-super cube root cube mean labeling of $AD(T_7)$ [triangle start from u_2] is shown in Figure 6.

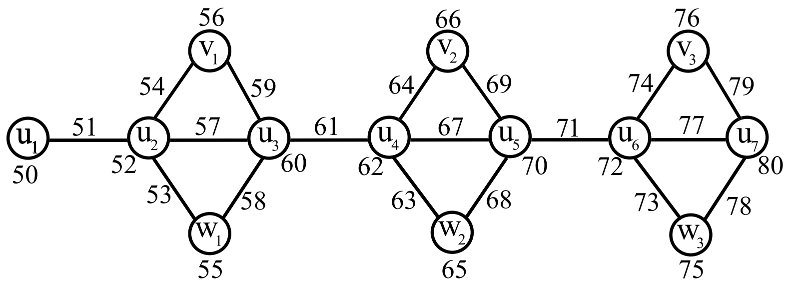

Figure 6: 50-super cube root cube mean labeling of $AD(T_7)$ [triangle start from u_2]

From the above cases, an alternate double triangular snake graph $AD(T_n)$ is a k-super cube root cube mean graph
□

Theorem 3.5. *Any Quadrilateral snake graph Q_n is a k-super cube root cube mean graph.*

Proof. Let Q_n be a quadrilateral snake graph.
Here p = 3n - 2 & q = 4n - 4 Hence p + q = 7n - 6.
Define a function $f : V(Q_n) \rightarrow \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$ by
$f(u_i)$ = k + 7i - 7, $1 \leq i \leq n$
$f(v_i)$ = k + 7i - 5, $1 \leq i \leq n - 1$

$$f(w_1) = \begin{cases} k+4, & k = 1,2,3; \\ k+5, & otherwise. \end{cases}$$

$f(w_i)$ = k + 7i - 2, $2 \leq i \leq n - 1$.

Then, the edge labels of Q_n are

$$f^*(u_1u_2) = \begin{cases} k+5, & k = 1,2,3; \\ k+4, & otherwise. \end{cases}$$

$f^*(u_iu_{i+1}) = k + 7i - 3,\ 2 \leq i \leq n - 1$
$f^*(u_iv_i) = k + 7i - 6,\ 1 \leq i \leq n - 1$
$f^*(u_{i+1}w_i) = k + 7i - 1,\ 1 \leq i \leq n - 1$
$f^*(v_iw_i) = k + 7i - 4,\ 1 \leq i \leq n - 1.$
Hence $f[V(Q_n)] \cup \{f^*(e) : e \in E(Q_n)\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}.$
Therefore any Quadrilateral snake graph Q_n is a k-super cube root cube

mean graph.
An example of 30-Super cube root cube mean labeling of Q_5 is shown in

Figure 7.

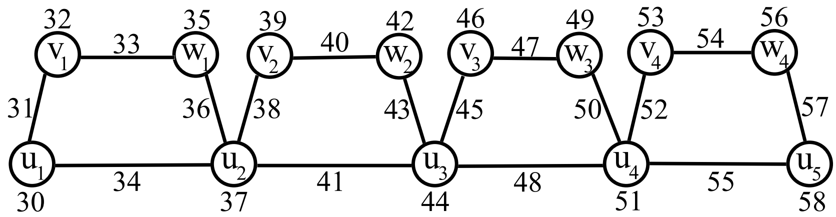

Figure 7: 30-Super cube root cube mean labeling of Q_5

□

Theorem 3.6. *Any alternate quadrilateral snake graph $A(Q_n)$ is a k-super cube root cube mean graph.*

Proof. Let A (Q_n) be an alternate quadrilateral snake graph.
In this theorem, consider two cases.

Case 1. Quadrilateral in $A(Q_n)$ starts from u_1.

In this case $p = \begin{cases} 2n, & if\ n\ is\ even; \\ 2n-1, & if\ n\ is\ odd. \end{cases}$

& $q = \begin{cases} \frac{5n-2}{2}, & if\ n\ is\ even; \\ \frac{5n-5}{2}, & if\ n\ is\ odd. \end{cases}$

Hence p + q = $\begin{cases} \frac{9n-2}{2}, & if\ n\ is\ even; \\ \frac{9n-7}{2}, & if\ n\ is\ odd. \end{cases}$

Define a function f: V(A(Q_n)) $\rightarrow$ {k, k + 1, k + 2, . . . , p + q + k − 1 } by

f(u_{2i-1}) = k + 9i - 9, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n+1}{2}$ if n is odd.
f(u_{2i}) = k + 9i - 2, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(v_i) = k + 9i - 7, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(w_1) = $\begin{cases} k+4, & if\ k = 1, 2, 3; \\ k+5, & otherwise. \end{cases}$

f(w_i) = k + 9i - 4, $2 \leq i \leq \frac{n}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

Then, the edge labels of A(Q_n) are

f*(u_1u_2) = $\begin{cases} k+5, & if\ k = 1, 2, 3; \\ k+4, & otherwise. \end{cases}$

f*($u_{2i-1}u_{2i}$) = k +9i -5, $2 \leq i \leq \frac{n}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

f*($u_{2i}u_{2i+1}$) = k +9i - 1, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*($u_{2i-1}v_i$) = k +9i -8, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*($u_{2i}w_i$) = k + 9i - 3, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*(v_iw_i) = k + 9i - 6, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
Hence f[V(A(Q_n))] ∪ {f*(e) : e ∈ E(A(Q_n))} = {k, k + 1, k + 2, . . . , p + q + k − 1}.
An example of 100-super cube root cube mean labeling of A(Q_6) [quadrilateral start from u_1] is shown in Figure 8.

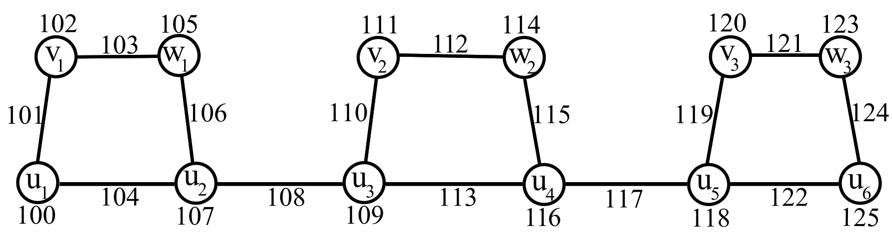

Figure 8: 100-super cube root cube mean labeling of $A(Q_6)$ [quadrilateral start from u_1]

Case 2. Quadrilateral in A(Q_n) starts from u_2

In this case p = $\begin{cases} 2n-2, & if\ n\ is\ even; \\ 2n-1, & if\ n\ is\ odd. \end{cases}$

& q = $\begin{cases} \frac{5n}{2}-4, & if\ n\ is\ even; \\ \frac{5n-5}{2}, & if\ n\ is\ odd. \end{cases}$

Hence p + q = $\begin{cases} \frac{9n-12}{2}, & if\ n\ is\ even; \\ \frac{9n-7}{2}, & if\ n\ is\ odd. \end{cases}$

Define a function f : V(A(Q_n))→ {k, k + 1, k + 2, . . . , p + q + k − 1 } by

f(u_{2i-1}) = k + 9i - 9, 1 ≤ i ≤ $\frac{n}{2}$ if n is even & 1 ≤ i ≤ $\frac{n+1}{2}$ if n is odd.
f(u_{2i}) = k + 9i - 7, 1 ≤ i ≤ $\frac{n}{2}$ if n is even & 1 ≤ i ≤ $\frac{n-1}{2}$ if n is odd. f(v_i) = k + 9i - 5, 1 ≤ i ≤ $\frac{n-2}{2}$ if n is even & 1 ≤ i ≤ $\frac{n-1}{2}$ if n is odd.

f(w_1) = $\begin{cases} k+6, & if\ k=1; \\ k+7, & otherwise. \end{cases}$

f(w_i) = k + 9i - 2, 2 ≤ i ≤ $\frac{n-2}{2}$ if n is even & 2 ≤ i ≤ $\frac{n-1}{2}$ if n is odd.

Then, the edge labels of A(Q_n) are

f*($u_{2i-1}u_{2i}$) = k +9i -8, 1 ≤ i ≤ $\frac{n}{2}$ if n is even & 1≤ i ≤ $\frac{n-1}{2}$ if n is odd.

f*(u_2u_3) = $\begin{cases} k+7, & if\ k=1; \\ k+6, & otherwise. \end{cases}$

f*($u_{2i}u_{2i+1}$) = k +9i - 3, 2 ≤ i ≤ $\frac{n-2}{2}$ if n is even & 2 ≤ i ≤ $\frac{n-1}{2}$ if n is

odd.
f*($u_{2i}v_i$) = k +9i -6, 1 ≤ i ≤ $\frac{n-2}{2}$ if n is even & 1 ≤ i ≤ $\frac{n-1}{2}$ if n is odd.
f*($u_{2i+1}w_i$) = k + 9i - 1, 1 ≤ i ≤ $\frac{n-2}{2}$ if n is even & 1 ≤ i ≤ $\frac{n-1}{2}$ if n is odd.
f*(v_iw_i) = k + 9i - 4, 1 ≤ i ≤ $\frac{n-2}{2}$ if n is even & 1 ≤ i ≤ $\frac{n-1}{2}$ if n is odd.
Hence f[V(A(Q_n))] ∪ {f*(e) : e ∈ E(A(Q_n))}={k, k + 1, k + 2, . . . , p + q + k − 1}.

An example of 100-super cube root cube mean labeling of $A(Q_7)$ [quadrilateral start from u_2] is shown in Figure 9.

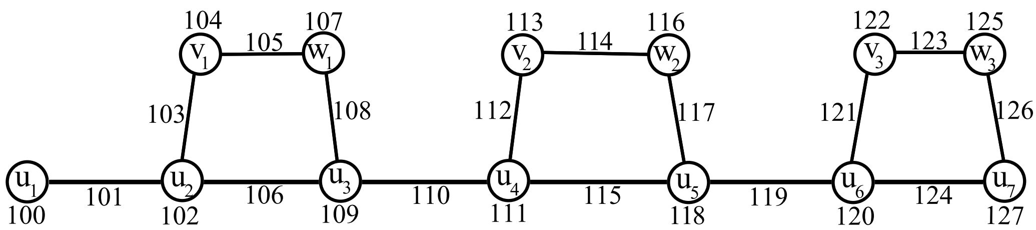

Figure 9: 100-super cube root cube mean labeling of $A(Q_7)$ [quadrilateral start from u_2]

From the above cases, an alternate quadrilateral snake graph $A(Q_n)$ is a k-super cube root cube mean graph.

□

Theorem 3.7. *Any double quadrilateral snake graph $D(Q_n)$ is a k-super cube root cube mean graph.*

Proof. Let $D(Q_n)$ be a double quadrilateral snake graph.
Here p = 5n - 4 & q = 7n - 7
Hence p + q = 12n - 11.
Define a function $f: V(D(Q_n)) \rightarrow \{k, k+1, k+2, \ldots, p+q+k-1\}$ by
$f(u_i) = k + 12i - 12, 1 \leq i \leq n$
$f(v_i) = k + 12i - 10, 1 \leq i \leq n - 1$
$f(w_i) = k + 12i - 7, 1 \leq i \leq n - 1$
$f(v'_i) = k + 12i - 6, 1 \leq i \leq n - 1$

$$f(w'_1)=\begin{cases} k+8, & if\ k=1,2,3,\ldots,9; \\ k+10, & otherwise. \end{cases}$$

$f(w'_i) = k + 12i - 2, 2 \leq i \leq n - 1.$

Then, the edge labels of $D(Q_n)$ are

$$f^*(u_1u_2) = \begin{cases} k+9, & if\ k=1,2,3,\ldots,9; \\ k+7, & otherwise. \end{cases}$$

$f^*(u_iu_{i+1}) = k + 12i - 5, 2 \leq i \leq n - 1$

$f^*(u_iv_i) = k + 12i - 11, 1 \leq i \leq n - 1$

$f^*(u_2w_1) = \begin{cases} k+10, & if\ k = 1,2,3,\ldots,9; \\ k+9, & otherwise. \end{cases}$

$f^*(u_{i+1}w_i) = k + 12i - 3, 2 \leq i \leq n - 1$

$f^*(v_iw_i) = k + 12i - 9, \ 1 \leq i \leq n - 1$
$f^*(u_iv'_i) = k + 12i - 8, \ 1 \leq i \leq n - 1$
$f^*(u_{i+1}w'_i) = k + 12i - 1, \ 1 \leq i \leq n - 1$
$f^*(v'_1w'_1) = \begin{cases} k+7, & if\ k = 1,2,3,\ldots,9; \\ k+8, & otherwise. \end{cases}$
$f^*(v'_iw'_i) = k + 12i - 4, 2 \leq i \leq n - 1$
Hence $f[V(D(Q_n))] \cup \{f^*(e) : e \in E(D(Q_n))\} = \{k, \ k+1, \ k+2, \ldots, \ p+q+k-1\}$.
Therefore any double quadrilateral snake graph $D(Q_n)$ is a k-super cube root cube mean graph.
An example of 25- Super cube root cube mean labeling of $D(Q_5)$ is shown

in Figure 10.

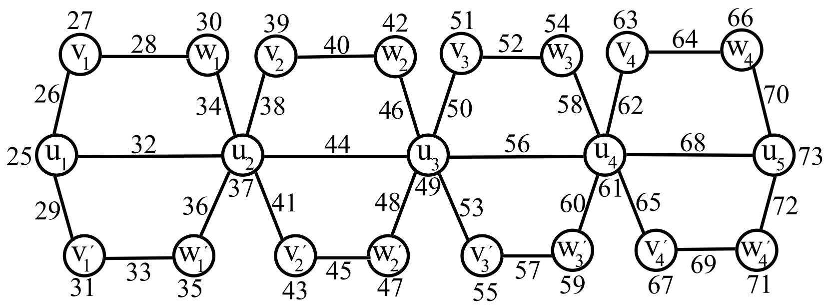

Figure 10: caption25- Super cube root cube mean labeling of $D(Q_5)$

□

Theorem 3.8. *Any alternate double quadrilateral snake graph $AD(Q_n)$ is a k-super cube root cube mean graph.*

Proof. Let AD(Q_n) be an alternate double quadrilateral snake graph In this theorem, consider two cases.

Case 1. Quadrilateral in AD(Q_n) starts from u_1

In this case p = $\begin{cases} 3n, \ if\ n\ is\ even; \\ 3n-2, \ if\ n\ is\ odd. \end{cases}$

& q = $\begin{cases} 4n-1, \ if\ n\ is\ even; \\ 4n-4, \ if\ n\ is\ odd. \end{cases}$

Hence p + q = $\begin{cases} 7n-1, \ if\ n\ is\ even; \\ 7n-6, \ if\ n\ is\ odd. \end{cases}$

Define a function f : V(AD(Q_n)) $\rightarrow$ {k, k + 1, k + 2, . . . , p + q + k − 1 }

by

f(u_{2i-1}) = k + 14i - 14, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n+1}{2}$ if n is odd.
f(u_{2i}) = k + 14i - 2, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(v_i) = k + 14i - 12, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(w_i) = k + 14i - 9, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(v'_i) = k + 14i - 8, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

f(w'_1) = $\begin{cases} k+8, \ if\ k = 1, 2, 3, \ldots, 9; \\ k+10, \ otherwise. \end{cases}$

f(w'_i) = k + 14i - 4, $2 \leq i \leq \frac{n}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

Then, the edge labels of AD(Q_n) are

f*(u_1u_2) = $\begin{cases} k+9, \ if\ k = 1, 2, \ldots, 9; \\ k+7, \ otherwise. \end{cases}$

f*($u_{2i-1}u_{2i}$) = k +14i -7, $2 \leq i \leq \frac{n}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

f*($u_{2i}u_{2i+1}$) = k +14i - 1, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*($u_{2i-1}v_i$) = k +14i -13, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

f*(u_2w_1) = $\begin{cases} k+10, \ if\ k = 1, 2, \ldots, 9; \\ k+9, \ otherwise. \end{cases}$

$f^*(u_{2i}w_i) = k + 14i - 5$, $2 \leq i \leq \frac{n}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

$f^*(v_iw_i) = k + 14i - 11$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i-1}v'_i) = k + 14i - 10$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
$f^*(u_{2i}w'_i) = k + 14i - 3$, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

$$f^*(v'_1w'_1) = \begin{cases} k+7, & if\ k = 1, 2, \ldots, 9; \\ k+8, & otherwise. \end{cases}$$

$f^*(v'_iw'_i) = k + 14i - 6$, $2 \leq i \leq \frac{n}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

Hence $f[V(AD(Q_n))] \cup \{f^*(e) : e \in E(AD(Q_n))\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.

An example of 115-super cube root cube mean labeling of $AD(Q_6)$ [quadrilateral start from u_1] is shown in Figure 11.

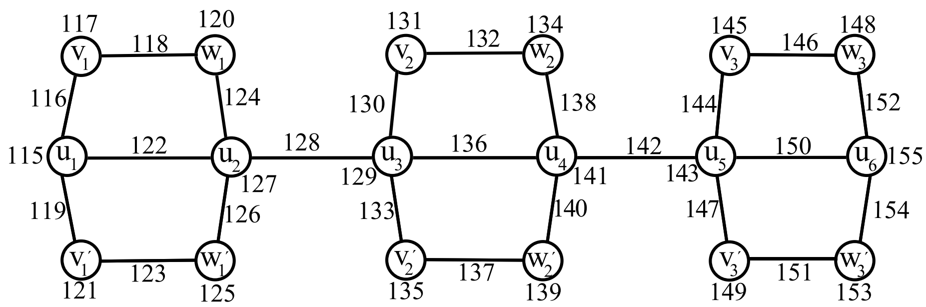

Figure 11: 115-super cube root cube mean labeling of $AD(Q_6)$ [quadrilateral start from u_1]

Case 2. Quadrilateral in $AD(Q_n)$ starts from u_2

In this case $p = \begin{cases} 3n-4, & if\ n\ is\ even; \\ 3n-2, & if\ n\ is\ odd. \end{cases}$

& $q = \begin{cases} 4n-7, & if\ n\ is\ even; \\ 4n-4, & if\ n\ is\ odd. \end{cases}$

Hence $p + q = \begin{cases} 7n-11, & if\ n\ is\ even; \\ 7n-6, & if\ n\ is\ odd. \end{cases}$

Define a function f:V(AD(Q_n)) $\rightarrow$ {k, k + 1, k + 2, . . . , p + q + k − 1 } by

f(u_{2i-1}) = k + 14i - 14, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n+1}{2}$ if n is odd.
f(u_{2i}) = k + 14i - 12, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(v_i) = k + 14i - 10, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(w_i) = k + 14i - 7, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f(v'_i) = k + 14i - 6, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

$$f(w'_1) = \begin{cases} k+11, & if\ k = 1,2,3,\ldots,7; \\ k+12, & otherwise. \end{cases}$$

f(w'_i) = k + 14i - 2, $2 \leq i \leq \frac{n-2}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

Then, the edge labels of AD(Q_n) are
f*($u_{2i-1}u_{2i}$) = k +14i -13, $1 \leq i \leq \frac{n}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

$$f^*(u_2u_3) = \begin{cases} k+10, & if\ k = 1,2,\ldots,7; \\ k+9, & otherwise. \end{cases}$$

f*($u_{2i}u_{2i+1}$) = k +14i - 5, $2 \leq i \leq \frac{n-2}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*($u_{2i}v_i$) = k +14i -11, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

$$f^*(u_3w_1) = \begin{cases} k+12, & if\ k = 1,2,\ldots,7; \\ k+11, & otherwise. \end{cases}$$

f*($u_{2i+1}w_i$) = k + 14i - 3, $2 \leq i \leq \frac{n-2}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*(v_iw_i) = k + 14i - 9, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*($u_{2i}v'_i$) = k +14i -8, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.
f*($u_{2i+1}w'_i$) = k + 14i - 1, $1 \leq i \leq \frac{n-2}{2}$ if n is even & $1 \leq i \leq \frac{n-1}{2}$ if n is odd.

$$f^*(v'_1w'_1) = \begin{cases} k+9, & if\ k = 1,2,\ldots,7; \\ k+10, & otherwise. \end{cases}$$

f*($v'_iw'_i$) = k + 14i - 4, $2 \leq i \leq \frac{n-2}{2}$ if n is even & $2 \leq i \leq \frac{n-1}{2}$ if n is odd.

Hence f[V(AD(Q_n))] ∪ {f*(e) : e ∈ E(AD(Q_n))} = {k, k + 1, k + 2, . . . , p + q + k − 1}.
An example of 115-super cube root cube mean labeling of AD(Q_7) [quadri-

lateral start from u_2] is shown in Figure 12.

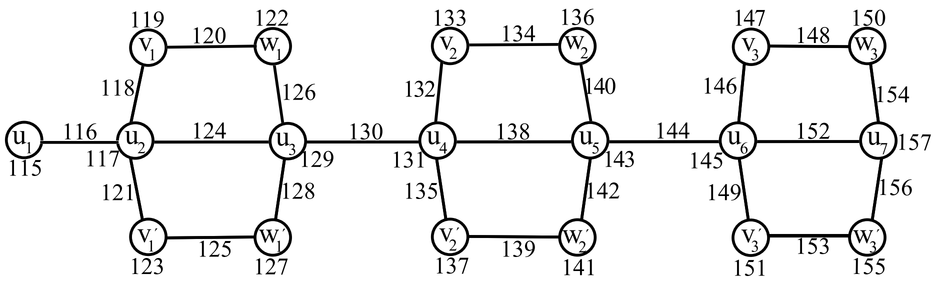

Figure 12: 115- super cube root cube mean labeling of $AD(Q_7)$ [quadrilateral start from u_2]

From the above cases, an alternate double quadrilateral snake graph $AD(Q_n)$ is a k-super cube root cube mean graph.

□

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